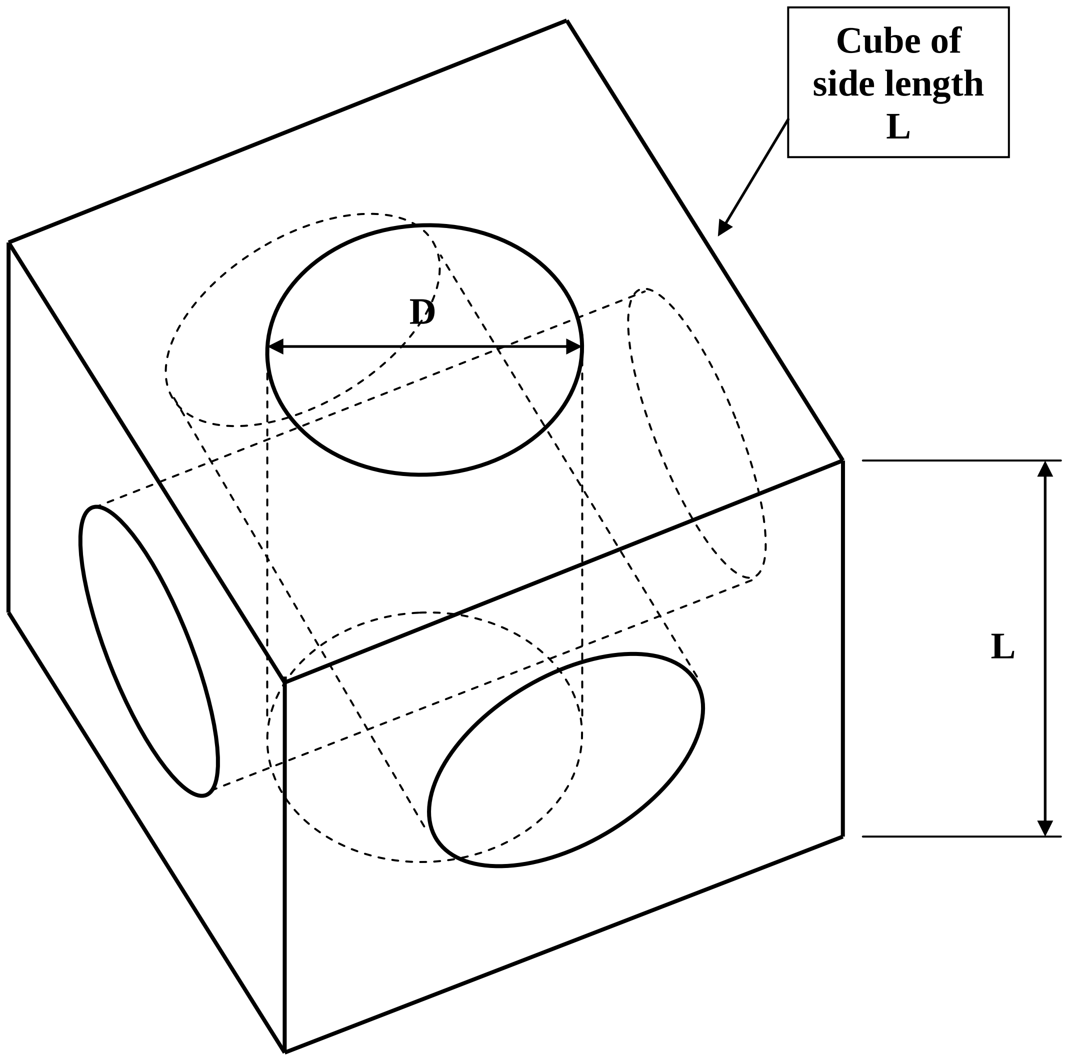
TO CALCULATE THE VOLUME OF A DRILLED CUBE

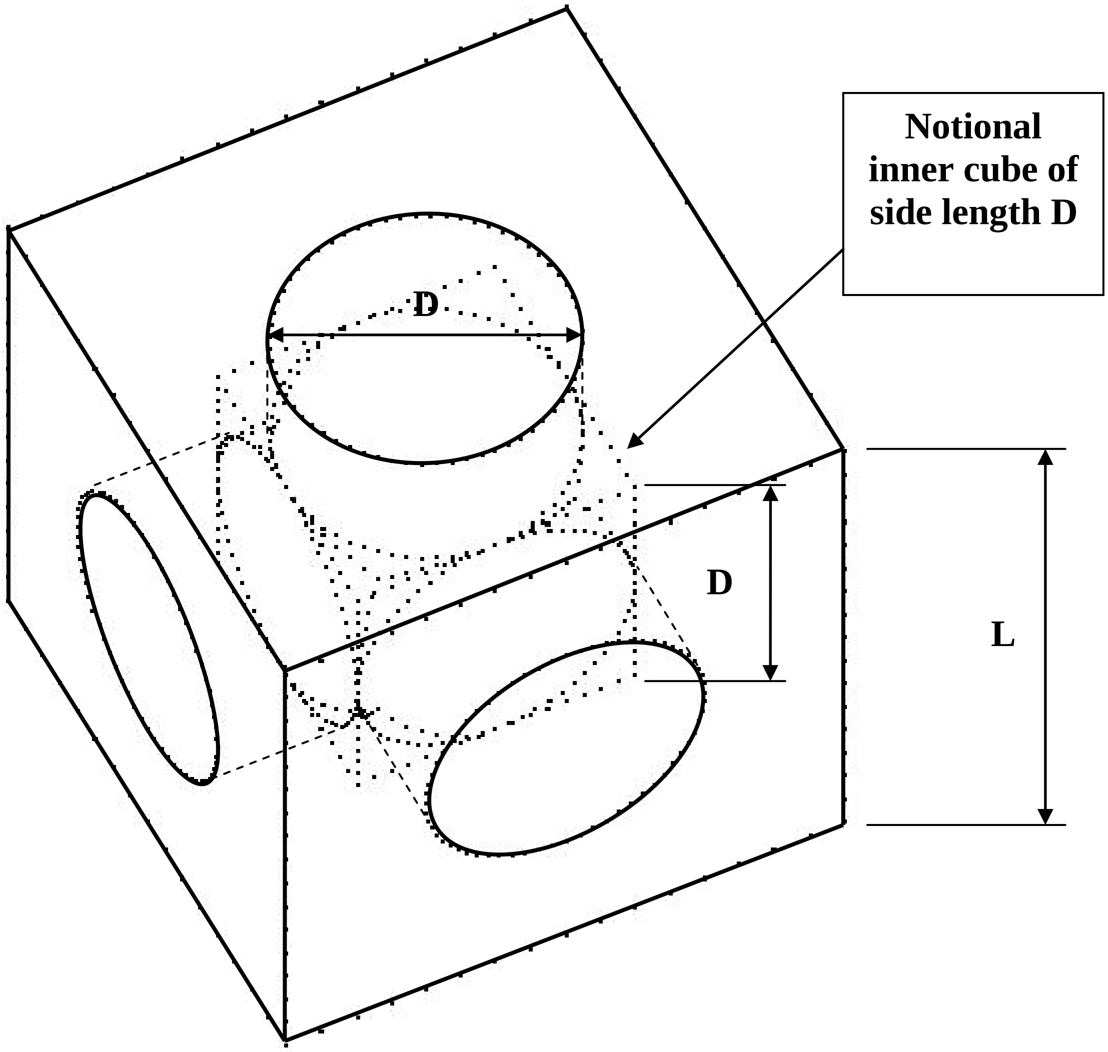
Problem Definition:

For a solid cube of side length, L, drill three mutually perpendicular holes of diameter D, through the centre of the cube and normal to the cube faces.

Calculate the remaining solid volume of the drilled cube.



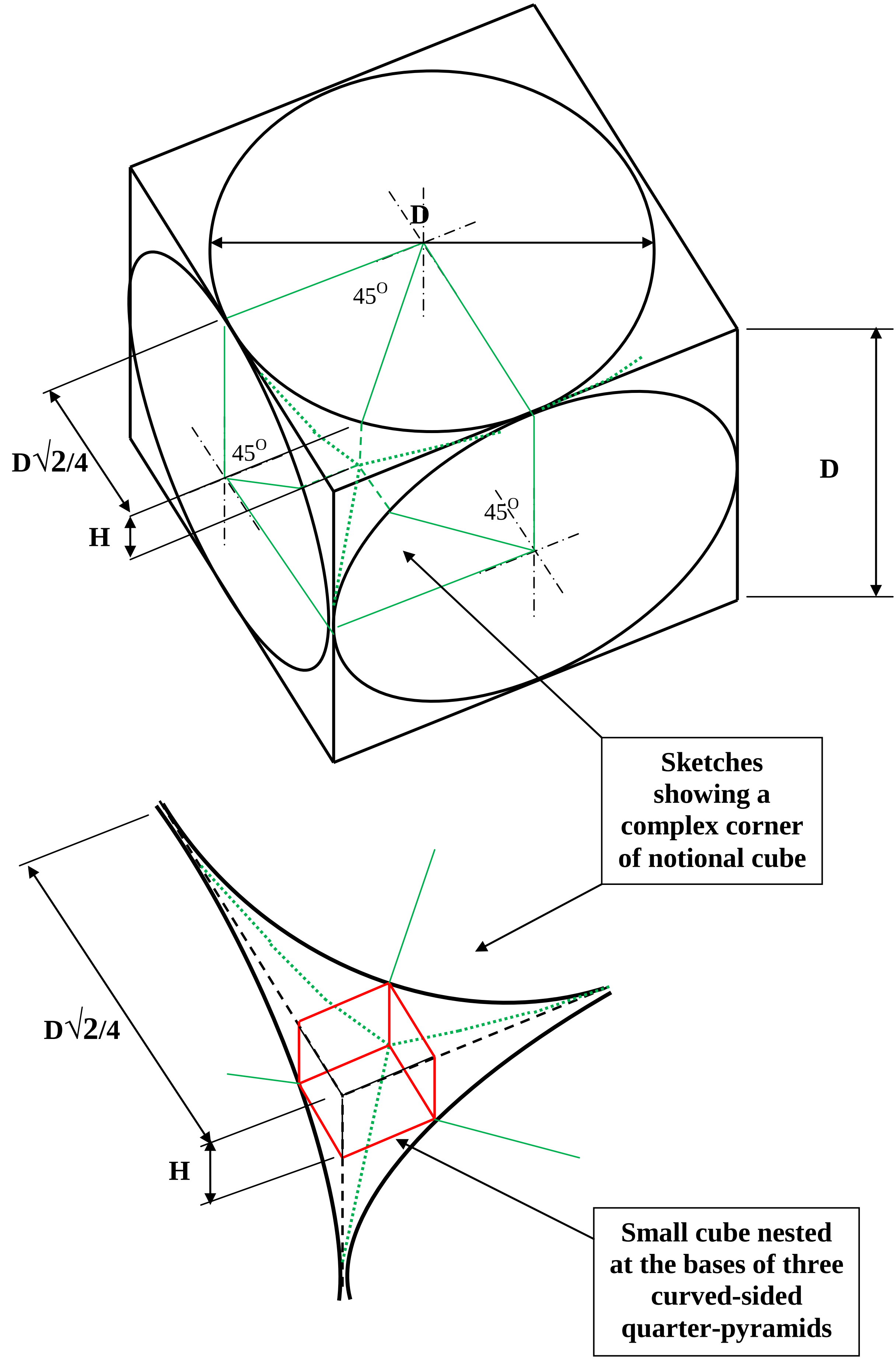
The above sketch represents a schematic view of the cube with three mutually perpendicular holes drilled through, but not showing any internal detail of the three intersecting holes.



From the above sketch, it can be seen that the intersection zone of the three drilled holes can be considered to be bounded by a notional inner cube of side length D.

At each of the eight corners of this notional cube, a complex solid shape is generated between the cube corner and the three adjacent intersecting cylindrical surfaces of the drilled holes.

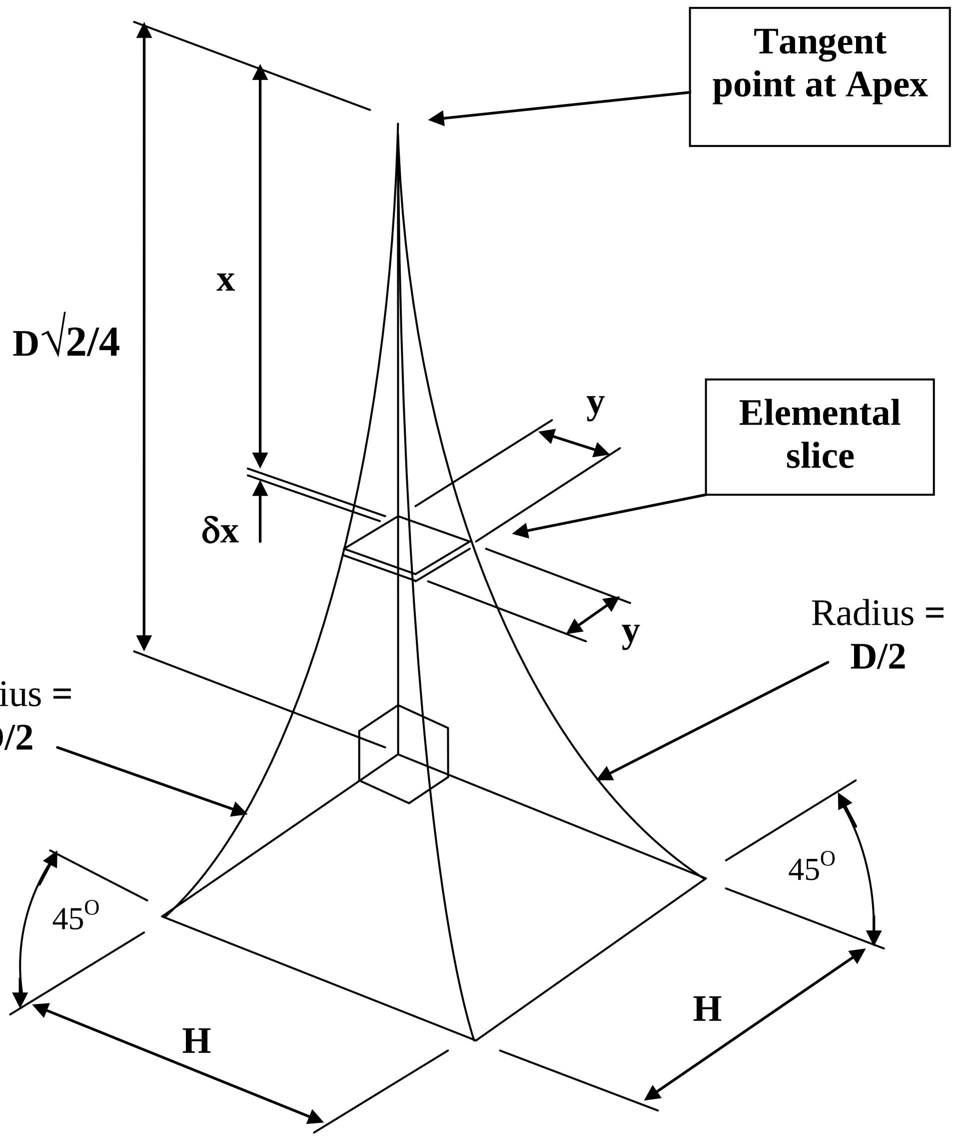
This complex shape can be considered to consist of three mutually perpendicular curved-sided quarter-pyramids whose square bases also form three adjacent sides of a small nested cube, as shown in the following two sketches.


D
45°
45°
45°
D√2/4
H
D
Sketches showing a complex corner of notional cube
D√2/4
H
Small cube nested at the bases of three curved-sided quarter-pyramids

These component volumes are quite easily calculated, with the exception of the curved-sided quarter-pyramid, whose volume can be calculated using classical Newtonian integration.

The length of base, H of these square-based curved-sided quarter-pyramids is given by, $H = (D/2) \times (1 - \cos 45^0) = (D/2) \times (1 - \sqrt{2}/2)$, and the perpendicular height of each pyramid above its base = $D\sqrt{2}/4$.

The following sketch facilitates the calculation of the volume of a curved-sided quarter-pyramid:



$H = (D/2) \times (1 - \sqrt{2}/2)$ $\qquad$ $y = (D/2) - \sqrt{[D^2/4 - x^2]}$

Volume of curved-sided quarter-pyramid

$$= \int_0^{D\sqrt{2}/4} y^2.dx = \int_0^{D\sqrt{2}/4} \left(D^2/2 - D\sqrt{[D^2/4 - x^2]} - x^2\right).dx$$

The following standard integral is required for the solution of the second term of the above compound integral:

$$\int [(a^2 - x^2)]^{0.5} .dx = (x/2).(a^2 - x^2)^{0.5} + (a^2/2).Sin^{-1}[x/a]$$

By substituting the above standard integral, the required volume

$$= \left[D^2.(x/2) - \left(D.(x/2).\sqrt{(D^2/4 - x^2)} + (D^3/8).Sin^{-1}[2x/D] \right) - (x^3/3)\right]_0^{D\sqrt{2}/4}$$

$$= D^3. [11\sqrt{2} - 6 - 3\pi]/96$$

$= \mathbf{0.001370533533\ D^3}$ (approximately)

The above value of the volume thus obtained has also been checked for accuracy using Numerical integration, with resulting agreement between the two methods.
[See Appendix for BASIC listing]

Volume of a complex corner of notional inner cube

= 3.(Volume of one quarter pyramid) + Volume of one small nested cube

$$= 3D^3. [11\sqrt{2} - 6 - 3\pi]/96 + D^3.(10 - 7\sqrt{2})/32$$

$= \mathbf{3 \times [0.001370533533\ D^3] + 0.003140783313\ D^3}$ (approximately)

$= \mathbf{0.007252383912\ D^3}$ (approximately)

The remaining solid volume of the modified cube after drilling of the three through holes is equal to the original volume of the cube, minus the sum of the volumes of six holes, each of diameter D by (L-D)/2 long, minus the volume of the notional inner cube, plus the sum of the volumes of the eight complex corner shapes.

Volume of original cube = L^3

Volume of 6 holes = $6.(\pi.D^2/4).(L-D)/2$

Volume of notional inner cube = D^3

Volume of 8 complex corners
$= 8.[3.(D^3.(11\sqrt{2}-6-3\pi)/96) + D^3.(10-7\sqrt{2})/32] = 0.0580190707D^3$ (approx)

Therefore, the remaining solid volume of drilled cube

$$= L^3 - 6.(\pi.D^2/4).(L-D)/2 - D^3 + 8.[3.(D^3.(11\sqrt{2}-6-3\pi)/96) + D^3.(10-7\sqrt{2})/32]$$

$$= L^3 - 3L.(\pi.D^2/4) + D^3.[12\pi - 32 + 8.(11\sqrt{2}-6) + (10-7\sqrt{2})]/32$$

$$= \underline{L^3 - 3L.(\pi.D^2/4) + D^3.(\sqrt{2})} \qquad \text{(exactly)}$$

$$= \underline{L^3 - 2.35619449.L.D^2 + 1.414213562.D^3} \qquad \text{(approximately)}$$

For the special case where D = L/2, the remaining volume of drilled cube

$$= \underline{L^3.[1 + \sqrt{2}/8 - 3.\pi/16]} \qquad \text{(exactly)}$$

$$= \underline{0.5877280728.L^3} \qquad \text{(approximately)}$$

By **N. 0. Williams**
17th. January 2016

APPENDIX

BASIC program listing for numerical integration of curved-sided quarter-pyramid.

```
10  CLS : REM "Spandrl3.txt"
15  PRINT
20  INPUT "INPUT RADIUS OF HOLE FORMING SPANDREL"; R: REM [R=0.5 for D=1.0]
25 REM Input a Large Number of Sections for greater accuracy.
30  INPUT "INPUT NUMBER OF SECTIONS OF SPANDREL"; N: REM [Say, 10,000 sections]
35  ATOTAL = 0: VTOTAL = 0
40  REM ***********************************************************************
50 DIM H (N + 1): DIM A (N + 1): H(0) = 0: H(N) = R * (1 - ((2 ^ 0.5)/2)): C = R * (2 ^ 0.5)/2
60 FOR J = 1 TO (N-1)
70 H(J) = R - ((R ^ 2) - (C * J/N) ^ 2) ^ 0.5
80 A(J) = (H(J)) ^ 2
90 ASUM = ASUM + A(J)
100 NEXT J
110 REM summation of all section areas
120 ATOTAL = ASUM + ((H(N)) ^ 2)/2
130 REM summation of all section volumes
140 VTOTAL = ATOTAL * C/N
150 print: print: PRINT "TOTAL VOLUME OF SPANDREL = "; VTOTAL: PRINT
160 REM ***********************************************************************
200 END
```

Running the above basic program for 10,000 sections and D = 1, the resulting volume by numerical integration

= 0.001370533

This computed value very closely agrees with the value already obtained by classical integration.